2013 Excellence in Mathematics Contest
Team Project Level II
(Below Precalculus)

CHANDLER-GILBERT COMMUNITY COLLEGE

School Name:

Group Members:

Reference Sheet

Formulas and Facts

You may need to use some of the following formulas and facts in working through this project. You may not need to use every formula or each fact.

$A = bh$ Area of a rectangle	$C = 2l + 2w$ Perimeter of a rectangle	$A = \pi r^2$ Area of a circle
$C = 2\pi r$ Circumference of a circle	$A = \frac{1}{2}bh$ Area of a triangle	$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope
$a^2 + b^2 = c^2$ Pythagorean Theorem	5280 feet = 1 mile	3 feet = 1 yard
16 ounces = 1 pound	2.54 centimeters = 1 inch	$h = -4.9t^2 + v_0 t + h_0$ $h = -16t^2 + v_0 t + h_0$
1 kilogram = 2.2 pounds	1 meter = 39.3701 inches	1 gigabyte = 1000 megabytes
1 mile = 1609 meters	1 gallon = 3.8 liters	1 square mile = 640 acres
1 sq. yd. = 9 sq. ft	1 cu. ft. of water = 7.48 gallons	1 ml = 1 cu. cm.
$V = \pi r^2 h$ Volume of cylinder	$V = (Area\ of\ \ Base) \cdot height$ Volume	$V = \frac{4}{3}\pi r^3$ Volume of a sphere
Lateral SA = $2\pi \cdot r \cdot h$ Lateral surface area of cylinder	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Quadratic Formula	$\tan\theta = \frac{\sin\theta}{\cos\theta}$

This team project is adapted from:
Bartkovich, K. (2013) Modeling Fuel Efficiency: MPG or GPHM?, *Mathematics Teacher*, v.107, no. 1, August 2013, National Council of Teachers of Mathematics (NCTM), Reston, VA

The Team Project is a group activity in which the students are presented an open ended, problem situation relating to a specific theme. The team members are to solve the problems and write a narrative about the theme which answers all the mathematical questions posed. Teams are graded on accuracy of mathematical content, clarity of explanations, and creativity in their narrative. We encourage the use of a graphing calculator.

Part 1: Background

In the August 2013 issue of *Mathematics Teacher*, Kevin Bartkovich writes:

> *Since the OPEC oil embargo of the early 1970's, the United States has become ever more conscious of the fuel efficiency of passenger cars. The standard for measuring fuel efficiency in the U.S. has been miles per gallon (mpg), and Americans are constantly buffeted with mpg ratings through new car advertisements, "stickers" on the vehicles at car dealerships and news reports of the latest political debate regarding vehicle emissions. In the midst of all this information, the use of mpg tends to obscure the implications of fuel efficiency standards, leading to widespread misconception about the mathematics of mpg.*

> *This investigation will lead us to model fuel efficiency with gallons per hundred miles (gphm), and the advantages we find in this new model will illustrate why stickers for the 2013 model-year cars display gphm as well as mpg.*

In this Team Project, you will investigate this new model of fuel efficiency, gallons per hundred miles.

Part 2: Fuel Efficiency

1. ***Without*** performing any calculations, which of the following transactions do you think results in the greatest gasoline savings? Clearly state which transaction you think will result in the greatest gasoline savings and describe why you think so. Again, this is not a computation focused question but rather an examination of your intuition.

   (*a*) Trading in a car that gets 10 mpg for a car rated at 11 mpg

   (*b*) Trading in a car that gets 22 mpg for a car rated at 26 mpg

   (*c*) Trading in a car that gets 38 mpg for a car rated at 50 mpg

It is expected that many students will choose c) since we see the greatest increase in the mpg. This is really just an item to ease them into the rest of the project. However, we will judge responses based on the team's ability to clearly communicate their thinking even though we are not looking for computations at this stage.

Part 2 continues…

2. Now we will focus on computations: In each case in problem 1, calculate the gasoline savings that would result from a year's driving—typically, 12,000 miles. How much would this save (each situation) in annual carbon dioxide emissions if burning 1 gallon of gasoline emits 20.4 pounds of CO_2?

(*a*) Trading in a car that gets 10 mpg for a car rated at 11 mpg

(*b*) Trading in a car that gets 22 mpg for a car rated at 26 mpg

(*c*) Trading in a car that gets 38 mpg for a car rated at 50 mpg

As the table shows, the greatest gasoline savings occur in situation a) Trading in a car that gets 10 mpg for a car rated at 11 mpg.

Table 1 Calculations of Gas Savings for Problem 1

MPG	GPM	Gas per Year (gallons)	Difference (gallons)
10	0.1000	1200	109
11	0.0909	1091	
22	0.0455	545	83
26	0.0385	462	
38	0.0263	316	76
50	0.0200	240	

Annual Carbon Dioxide emmisions:

a) We would save 109 gallons so $109 \cdot 20.4 = 2223.6$ pounds of CO_2.

b) We would save 83 gallons so $83 \cdot 20.4 = 1693.2$ pounds of CO_2.

c) We would save 76 gallons so $76 \cdot 20.4 = 1550.4$ pounds of CO_2.

Part 2 continues…

3. Fill in the blanks so that all three transactions result in about the same gasoline savings. What is the savings in a year of driving (again assuming a car is driven 12,000 miles per year)? Clearly and neatly show all your work in the space provided.

(*a*) Trading in a car that gets 10 mpg for a car rated at 11 mpg

(*b*) Trading in a car that gets 16.5 mpg for a car rated at _____ mpg

(*c*) Trading in a car that gets _____ mpg for a car rated at 50 mpg

First, we see that situation a) yields a savings of $\frac{12000}{10} - \frac{12000}{11} = \frac{1200}{11}$ gallons.

Situation b)

$$\frac{12000}{16.5} - \frac{12000}{M} = \frac{1200}{11} \text{ gallons}$$
$$\frac{12000}{16.5} - \frac{1200}{11} = \frac{12000}{M}$$
$$\frac{6800}{11} = \frac{12000}{M}$$
$$6800M = 132000$$
$$M \approx 19.4$$

Trading in a car that gets 16.5 mpg for a car rated at <u>19.4</u> mpg

Situation c)

$$\frac{12000}{M} - \frac{12000}{50} = \frac{1200}{11} \text{ gallons}$$
$$\frac{12000}{M} = \frac{3840}{11}$$
$$3840M = 132000$$
$$M \approx 34.4$$

Trading in a car that gets <u>34.4</u> mpg for a car rated at 50 mpg

In each case, the savings is approximately 109.1 gallons of gasoline.

Part 3: Gallons per Hundred Miles (gphm)

1. Fill in the following table for miles per gallon vs. gallons per hundred miles. Clearly and neatly show all your computations in the space provided.

MPG	GPHM
10	10.0
20	5.0
25	4.0
40	2.5
50	2.0
80	1.25
100	1

The relationship between gphm and mpg is $gphm = \frac{100}{mpg}$.

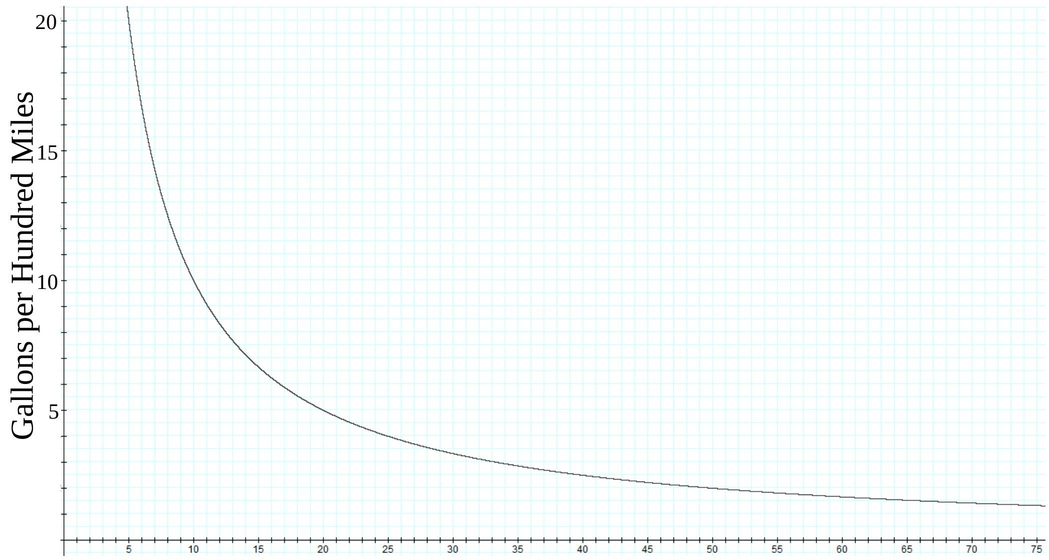
Judge the student work based on the mathematics shown.

Part 3 continues…

2. a) Find and write a general formula for the relationship between gphm and mpg. That is, if one was given the mpg rating, how would they compute gphm?

$$gphm = \frac{100}{mpg}$$

b) Sketch a graph of gphm as a function of mpg. Clearly label your axes. Explain how the graph confirms the findings in Part 2 (refer back to the situations in Part 2 where you compared increases in fuel efficiency).



Miles per Gallon

The number of gallons per hundred miles changes most significantly when the mpg rating is low. The number of gallons per hundred miles does not change significantly when the mpg rating is high. As seen in Part 2, an increase in the fuel efficiency dramatically reduces the number of gallons used (thus more efficient) when the efficiency rating is smaller.

3. If you want to calculate gasoline savings in a comparison of two cars, which rating is more helpful—mpg or gphm? Provide a written rationale for your response.

Using gphm allows for an easier comparison since we only need to make an additive comparison (subtract). In order to compare with mpg, we see that there is much more work to do and that the results are deceptive. Intuitively, it seems that greater fuel savings occur when increasing from, for example, 35 mpg to 45 mpg. Instead, we see greater fuel savings when increasing from, for example, 5 mpg to 15 mpg.

Part 4: Average Fuel Efficiency

1. When you drive to work, your hybrid car goes uphill for 20 miles, and you notice that the fuel efficiency during that time is 20 mpg. Returning home, you drive downhill for the same 20 miles, and your fuel efficiency is 100 mpg. What is your average mpg for the round-trip? Hint: The answer is ***NOT*** 60 mpg!

On the way to work, the car traveled 20 miles with a fuel efficiency of 20 mph. Therefore, 1 gallon of fuel was used.

One the way home from work, the car again traveled 20 miles with a fuel efficiency of 100 mpg. Therefore, $\frac{1}{5}$ gallon of fuel was used.

The total distance driven is 40 miles consuming $1\frac{1}{5}$ gallons of fuel.

The average mpg is $\dfrac{40 \text{ miles}}{1\frac{1}{5} \text{ gallons}} = 33\frac{1}{3}$ mpg.

2. If you have two cars, one of which is rated 14 mpg and the other 36 mpg, what is your average mpg, assuming that you drive each car the same number of miles? Clearly state any assumptions you make and let your mathematical work justify your solution.

I present a hypothetical student response.

Students may experiment by assuming that each car traveled a distance of 100 miles.

$$\frac{100 \text{ miles}}{14 \text{ mpg}} = \frac{50}{7} \text{ gallons} \approx 7.14 \text{ gallons}$$

$$\frac{100 \text{ miles}}{36 \text{ mpg}} = \frac{25}{9} \text{ gallons} \approx 2.78 \text{ gallons}$$

$$\text{Average mpg} \approx \frac{200 \text{ miles}}{7.14 + 2.78 \text{ gallons}} \approx 20.16 \text{ mpg}$$

Part 4 continues…

3. Suppose you have two cars. One is rated at M_1 mpg and the other at M_2 mpg. Suppose that you drive a total of D miles per year (with both cars) and each car is driven the same number of miles. Write a formula, in terms of only M_1 and M_2, for the average mpg. Clearly and neatly show all your computations in the space provided.

Car 1: $\dfrac{\frac{1}{2}D \text{ miles}}{M_1 \text{ mpg}} = G_1 \text{ gallons consumed}$

Car 2: $\dfrac{\frac{1}{2}D \text{ miles}}{M_2 \text{ mpg}} = G_2 \text{ gallons consumed}$

$$\text{Average mpg} = \frac{D}{G_1 + G_2} = \frac{D}{\frac{\frac{1}{2}D}{M_1} + \frac{\frac{1}{2}D}{M_2}}$$

$$= \frac{D}{\frac{1}{2}\left(\frac{D}{M_1} + \frac{D}{M_2}\right)} = \frac{2D}{\left(\frac{D}{M_1} + \frac{D}{M_2}\right)}$$

$$= \frac{2D}{\left(\frac{DM_2 + DM_1}{M_1 M_2}\right)} = \frac{2DM_1M_2}{DM_2 + DM_1}$$

$$= \frac{2M_1M_2}{M_2 + M_1}$$

Part 4 continues…

4. If you have two cars, one of which is rated 5 gphm and the other 2 gphm, what is your average gphm? Clearly state any assumptions you make and let your mathematical work justify your solution.

Students may assume that each car travels a distance of 100 miles. If so, Car 1 will consume 5 gallons of fuel. Car 2 will consume 2 gallons of fuel.

$$\text{average gphm} = \frac{5 \text{ gallons} + 2 \text{ gallons}}{2} = 3.5 \text{ gallons}$$

5. Suppose you have two cars. One is rated at G_1 gphm and the other at G_2 gphm. Suppose that you drive a total of D miles per year (with both cars) and each car is driven the same number of miles. Write a formula, in terms of only G_1 and G_2, for the average mpg. Clearly and neatly show all your computations in the space provided.

$$\text{average gphm} = \frac{G_1 + G_2}{2}$$

Part 4 continues…

6. Discuss the relative merits of the mpg rating and the gphm rating. In which situation is one more helpful than the other?

Answers will vary. Judge student's ability to make a claim and back it up with reasons.

Part 5: Improving Fuel Efficiency

1. What strategy would have a greater overall effect on carbon emissions—raising the efficiency of the 50-mpg vehicles by 10% or raising the efficiency of an equal number of 20-mpg vehicles by 10%? Provide a clear and coherent mathematical rationale for your response. Recall that 1 gallon of gasoline emits 20.4 pounds of CO_2.

$$\frac{1\text{ gallon}}{50\text{ miles}}\cdot\frac{20.4\text{ lbs}}{1\text{ gallon}}\approx 0.408\text{ lbs. per gallon}$$
$$\frac{1\text{ gallon}}{55\text{ miles}}\cdot\frac{20.4\text{ lbs}}{1\text{ gallon}}\approx 0.371\text{ lbs. per gallon}$$

By increasing the efficiency rating by 10%, from 50-mpg to 55-mpg, we see a decrease in carbon emissions of $0.408-0.371=0.37\text{ lbs. per gallon}$.

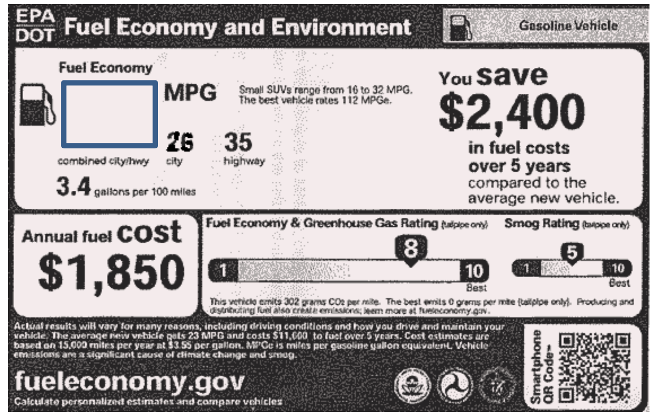
$$\frac{1\text{ gallon}}{20\text{ miles}}\cdot\frac{20.4\text{ lbs}}{1\text{ gallon}}\approx 1.02\text{ lbs. per gallon}$$
$$\frac{1\text{ gallon}}{22\text{ miles}}\cdot\frac{20.4\text{ lbs}}{1\text{ gallon}}\approx 0.927\text{ lbs. per gallon}$$

By increasing the efficiency rating by 10%, from 20-mpg to 22-mpg, we see a decrease in carbon emissions of $1.02-0.927=0.93\text{ lbs. per gallon}$.

Improving fuel efficiency by 10% of the 20-mpg vehicles has a greater impact on carbon emissions.

Part 5 continues…

2. The EPA calculates combined mpg by assuming 55% city driving and 45% highway driving. What is the combined mpg for a vehicle that is rated 26 mpg city and 35 mpg highway as shown in the figure? In other words, what number should go in the box which would normally show the combined city/highway fuel economy?



Earlier, we found the average mpg assuming that an equal distance was traveled at each mpg rating. In this case, 55% of the driving is at a rating of 26 mpg and 45% at a rating of 35 mpg.

$$
\begin{aligned}
\text{Average mpg} &= \frac{D}{G_1+G_2} = \frac{D}{\frac{0.55D}{M_1}+\frac{0.45D}{M_2}} \\
&= \frac{D}{\frac{0.55D}{26}+\frac{0.45D}{35}} \\
&= \frac{D}{\frac{35\cdot 0.55D+26\cdot 0.45D}{26\cdot 35}} \\
&= D\cdot\frac{26\cdot 35}{35\cdot 0.55D+26\cdot 0.45D} \\
&= \frac{26\cdot 35}{35\cdot 0.55+26\cdot 0.45} \\
&\approx 29.4\text{ mpg}
\end{aligned}
$$

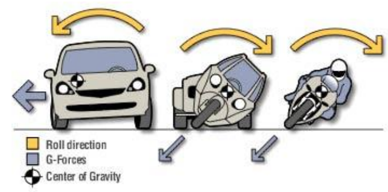
Typically, they report a whole number on these stickers so the number that should be placed in the box is 29.

Part 5 continues…

3. The website www.hybridcars.com provides the following background for a new car predicted to be available to the public sometime in 2014.

What do you get when you combine the exhilaration of riding a fast motorcycle, the safety and comfort of a commuter car, and the fuel efficiency of advanced automotive technologies? The VentureOne—a two-passenger, three-wheeled, 100-mpg plug-in series hybrid from Venture Vehicles in Los Angeles.

Venture compares the driving sensation of the V1 to "flying a jet fighter 2 feet off the ground." Capable of reaching top speeds of approximately 100 mph, it takes corners like a racing motorcycle that leans almost completely to one side. The two wheels and propulsion system in the back stay firmly on the ground, while the single front wheel and cabin—more like a glass-enclosed cockpit—tilts at angles up to 45 degrees. The automated tilting system, developed by Carver Engineering in the Netherlands and licensed by Venture Vehicles, uses a combination of hydraulic and mechanical technologies to determine the ideal angle and balance based on the traveling speed, rate of acceleration, and road conditions.



The VentureOne (recently renamed the Persu Hybrid), purportedly will attain a fuel efficiency rating of 100-mpg. Suppose this is the ***combined mpg*** (as described in the previous problem) and suppose that this vehicle will be rated at 70-mpg for city driving. Recall that the EPA calculates combined mpg by assuming 55% city driving and 45% highway driving. What would the highway driving mpg rating need to be to guarantee a combined mpg rating of 100-mpg? Mathematically justify your response.

Let $M_1 =$ city mpg rating and $M_2 =$ highway mpg rating

$$\text{Combined mpg} = \frac{M_1 \cdot M_2}{M_2 \cdot 0.55 + M_1 \cdot 0.45}$$

$$100 = \frac{70 \cdot M_2}{M_2 \cdot 0.55 + 70 \cdot 0.45}$$

$$100 = \frac{70 \cdot M_2}{M_2 \cdot 0.55 + 31.5}$$

$$100\left(M_2 \cdot 0.55 + 31.5\right) = 70 \cdot M_2$$

$$55M_2 + 3150 = 70 \cdot M_2$$

$$3150 = 15 \cdot M_2$$

$$M_2 = 210$$

The highway mpg rating would have to be 210-mpg in order for the combined rating to be 100-mpg with a city rating of 70-mpg.

Part 5 continues…

4. Again consider the situation where the VentureOne purportedly has a mpg rating of 100-mpg. Create a graph of the function $M_1 = f(M_2)$ where M_1 represents the city driving mpg rating and M_2 represents the highway driving mpg rating. That is, create the graph a function showing the relationship between the city mpg rating and the highway mpg rating. Recall that the EPA calculates combined mpg by assuming 55% city driving and 45% highway driving.

Let $M_1 =$ city mpg rating and $M_2 =$ highway mpg rating

$$\text{Combined mpg} = \frac{M_1 \cdot M_2}{M_2 \cdot 0.55 + M_1 \cdot 0.45}$$

$$100 = \frac{M_1 \cdot M_2}{M_2 \cdot 0.55 + M_1 \cdot 0.45}$$

$$100\left(M_2 \cdot 0.55 + M_1 \cdot 0.45\right) = M_1 \cdot M_2$$

$$55M_2 + 45M_1 = M_1 \cdot M_2$$

$$55M_2 - M_1 \cdot M_2 = -45M_1$$

$$M_2(55 - M_1) = -45M_1$$

$$M_2 = \frac{-45M_1}{55 - M_1} = \frac{45M_1}{M_1 - 55}$$